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AN INTERESTING PROPERTY OF THE KERR METRIC

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Abstract: We show that the Einstein tensor of the Kerr model can be represented in two parts, one originating from the static elliptical geometry and the second describing the rotational part, which is attached to the static part by an anholonomic transformation.

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1. INTRODUCTION

In earlier papers [1], we have systematically developed the Kerr model. We started from a flat elliptic-hyperbolic geometry and pulled up a surface on it in a 5-dimensional flat embedding space similar to the Flamm paraboloid of Schwarzschild geometry. Finally, we implemented rotational effects in the static model using an anholonomic transformation. The resulting model is known as the Kerr model, which describes the external field of a rotating stellar object. We have made the remarkable observation that the Einstein tensor of the static auxiliary metric and the part added by the anholonomic transformation are formally identical. It is evident that the elliptic-hyperbolic geometry already contains rotational elements. In the following sections, we aim to address this, whereby much is only sketched so that the actual problem is not relegated to the background by the large amount of formulae. For formal details on the calculations of the field quantities and Einstein's field equations, refer to [1] for a detailed description of the Kerr problem.

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2. THE STRUCTURE OF THE KERR MODEL

The Kerr model is based on an axially symmetrical solution of Einstein's field equations. It describes the field of a rotating stellar object. By tradition, the line element of the model is written in Boyer-Lindquist coordinates. However, we prefer a different notation because, in the Boyer-Lindquist form, variables are summarized as quantities to which no geometric meaning can be assigned.

We base the metric exclusively on quantities that have a clear geometric meaning. For example, the line element of an oblate ellipsoid of revolution in polar coordinates is $\{r, \vartheta, \varphi\}$

$$ds^2 = a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 . \tag{2.1}$$

Here is

$$a_R = \frac{\Lambda}{A}, \quad \Lambda^2 = r^2 + a^2\cos^2\vartheta, \quad A^2 = r^2 + a^2, \quad \sigma = A\sin\vartheta , \tag{2.2}$$

with 'a' representing the linear eccentricity of the ellipses, Λ the geometric mean, and A denoting the arithmetic mean of the focal rays of the ellipses. σ is the distance of a point on an ellipse from the axis of symmetry of the ellipsoid. We refer to a_R as the elliptic parameter. If we set $a = 0$, we obtain a spherical line element from (2.1).

As we will show later, the geometry of the ellipsoid already contains elements that refer to the rotational model. The elliptical factor allows the following reformulation:

$$a_R^2 = \frac{\Lambda^2}{A^2} = 1 - \frac{a^2}{A^2}\sin^2\vartheta = 1 - \omega^2\sigma^2, \quad \omega = \frac{a}{A^2} . \tag{2.3}$$

Here, ω is the angular velocity of an observer following the rotational forces of the Kerr field[1].

The ellipses and hyperbolae have curvature vectors

$$\{\rho_E, 0, 0\}, \quad \{0, \rho_H, 0\}, \quad \rho_E = \frac{\Lambda^3}{rA}, \quad \rho_H = -\frac{\Lambda^3}{a^2\sin\vartheta\cos\vartheta} . \tag{2.4}$$

The angles of ascent of the curvature vectors are represented by $\theta_E = \theta_E(r, \vartheta)$ and $\theta_H = \theta_H(r, \vartheta)$. Since the elliptic and hyperbolic curves are orthogonal trajectories, $d\theta_E = -d\theta_H \overset{*}{=} d\theta$ applies. With the relations

$$\sin\theta = \frac{r}{\Lambda}\sin\vartheta, \quad \cos\theta = \frac{A}{\Lambda}\cos\vartheta , \tag{2.5}$$

the derivatives of the angle θ can be calculated. We exclusively use the 3-beine

$$\partial_\alpha = \underset{\alpha}{e}{}^i \partial_i, \quad \partial_1 = \frac{1}{a_R}\frac{\partial}{\partial r}, \quad \partial_2 = \frac{1}{\Lambda}\frac{\partial}{\partial\vartheta}, \quad \partial_3 = \frac{1}{\sigma}\frac{\partial}{\partial\varphi} . \tag{2.6}$$

By differentiating (2.5), we finally obtain

[1] $\alpha_R = 1/a_R$ is the Lorentz factor of the orbital motion

$$\theta_{|\alpha} = \left\{ -\frac{1}{\rho_H}, \frac{1}{\rho_E}, 0 \right\} \tag{2.7}$$

and

$$\rho_{E|1} = 1 - \rho_E^2 \tilde{\Omega}^{3\alpha} \tilde{\Omega}_{\alpha 3}, \quad \rho_{H|2} = 1 + \rho_H^2 \tilde{\Omega}^{3\alpha} \tilde{\Omega}_{\alpha 3}, \tag{2.8}$$

where the quantities $\tilde{\Omega}$ are rotational, which will be discussed in detail. First, we will discuss how they are generated.

The elliptical and hyperbolic curves both have evolutes; they themselves are the evolvents of these evolutes. The curvature vectors (2.4) are normal to the evolvents in each case but are tangent to the evolutes. More precisely, the starting point of the curvature vectors lies on the evolutes. Together with the evolvents, they form a polar coordinate system, whereby it should be noted that the poles are not fixed but move on the evolutes when the tips of the curvature vectors move on the evolvents. This movement results in rotational contributions due to the changes in the radii of curvature, according to (2.8).

We use the Ricci-rotation coefficients to formulate the field quantities. Their components are the curvature quantities, i.e., the reciprocal values of the radii of curvature. If we read off the 3-beine from (2.1), considering (2.2), and use them to calculate the Ricci-rotation coefficients, we have the following components for the curvatures of the ellipses, hyperbolae, and circles of the ellipsoids of revolution

$$B_\alpha = \left\{ \frac{1}{\rho_E}, 0, 0 \right\}, \quad N_\alpha = \left\{ 0, \frac{1}{\rho_H}, 0 \right\}, \quad C_\alpha = \left\{ \frac{1}{\sigma}\sin\theta, \frac{1}{\sigma}\cos\theta, 0 \right\}. \tag{2.9}$$

These expressions have to be differentiated to calculate the field quantities. From (2.8), it follows that rotational terms occur in the Einstein tensor. We will now deal with these in more detail.

Here, we must anticipate the rotational properties of the Kerr model. We have already stated in (2.3) that the elliptical geometry provides an expression for the orbital velocity of a rotating observer in the Kerr field. According to (2.3), the angular velocity $\omega = \omega(r)$ depends on the distance from the axis of symmetry in such a way that the angular velocity becomes $\omega \to 0$ for $r \to \infty$. Therefore, we have a differential rotation law with the convenient property that the orbital velocity decreases outwardly and vanishes at infinity.

Considering the last equation in (2.2),

$$(\omega\sigma)_{|\alpha} = \omega\sigma_{|\alpha} + \omega_{|\alpha}\sigma, \quad \omega_{|2} = 0,$$

two quantities can be designed

$$H_{\alpha 3} = i\alpha_R^2 \omega \sigma_{|\alpha}, \quad D_{\alpha 3} = i\alpha_R^2 \omega_{|\alpha} \sigma. \tag{2.10}$$

The first is the relativistic generalization of the Coriolis force; the second is a shear force. The latter is formed because observers closer to the inside are faster than those on the outside. This means that such observers slide each other by.

The definitions (2.2) result in

$$\omega_{|\alpha}\sigma = -2\omega\sigma_{|\alpha} \tag{2.11}$$

which means that $D_{13} \stackrel{*}{=} -2H_{13}, D_{31} = 0$. This numerical relation simplifies the calculations but hides the geometric content. Finally, we obtain the quantities

$$\begin{aligned} &\tilde{\Omega}_{\alpha 3} = \left\{\tilde{H}_{13}, \tilde{H}_{23}\right\}, \quad \tilde{\Omega}_{3\alpha} = \left\{\tilde{H}_{13}, -\tilde{H}_{23}\right\} \\ &\tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{3\alpha} = \tilde{H}_{13}\tilde{H}_{13} - \tilde{H}_{23}\tilde{H}_{23}, \quad \tilde{\Omega}^{\alpha 3}\tilde{\Omega}_{\alpha 3} = \tilde{H}_{13}\tilde{H}_{13} + \tilde{H}_{23}\tilde{H}_{23} \end{aligned}, \tag{2.12}$$

where, from now on, we will use tildes to indicate that the quantities mentioned refer to the flat auxiliary geometry.

We must add a time-like element to the metric (2.1) using the original Minkowski notation $x^4 = i(c)t$ to get closer to the actual Kerr metric. Furthermore, on the $\{r, \vartheta\}$-section of the ellipsoid of revolution, we pull up a surface that has an elliptical cross-section and is reduced to Flamm's paraboloid if we set the rotation parameter $a = 0$, and on the $\{r, \varphi\}$-section, we pull up a surface with a circular cross-section.

Together, the surfaces result in the Kerr surface, which is very close to the Kerr geometry. According to [1], the metric has the form

$$ds^2 = \alpha_S^2 a_R^2 dr^2 + \Lambda^2 d\vartheta^2 + \sigma^2 d\varphi^2 + a_S^2 (idt)^2, \tag{2.13}$$

where the newly occurring factor is given by

$$\alpha_S = \frac{A}{\delta}, \quad a_S = \alpha_S^{-1}, \quad \delta^2 = r^2 + a^2 - 2Mr. \tag{2.14}$$

Reading the 4-beine from (2.13), using it to calculate the Ricci-rotation coefficients and finally the Ricci, we obtain a collection of terms on the right-hand side of the Einstein tensor, which are difficult to understand and form the stress-energy-momentum tensor of the auxiliary model. We prefer a different procedure based on the possibility of embedding the Kerr geometry in a 5-dimensional flat space.

For the 5-dimensional field quantities, we set

$$\begin{aligned} &B_a = \left\{\sin\varepsilon \frac{1}{\rho_E}, \cos\varepsilon \frac{1}{\rho_E}, 0, 0, 0\right\}, \quad N_a = \left\{0, 0, \frac{1}{\rho_H}, 0, 0\right\} \\ &C_a = \left\{\sin\varepsilon \sin\theta, \cos\varepsilon \sin\theta, \cos\theta, 0, 0\right\} \frac{1}{\sigma}\sigma_{|a} \\ &E_a = \left\{-\frac{1}{\rho_S}, \frac{1}{\rho_S}\tan\varepsilon, 0, 0, 0\right\} = \left\{\cos\varepsilon, -\sin\varepsilon, 0, 0, 0\right\}\left(-\frac{1}{\rho_S \cos\varepsilon}\right) \end{aligned}. \tag{2.15}$$

The index sequence is $a = \{0, 1, ..., 4\}$. The 0-th dimension is the extra dimension. E is the force of gravity, which is calculated from the factor a_S of the metric (2.13). Furthermore, the radius of curvature of the time-like pseudocircle $\rho_S \cos\varepsilon$ is defined with

$$\rho_S = \Lambda\sqrt{\frac{2r}{M}}\Phi^2, \quad \Phi^2 = \frac{r^2 + a^2}{r^2 - a^2}.$$

The new angular functions in (2.15) have the meaning

$$\cos\varepsilon = a_S = \alpha_S^{-1}, \quad \sin\varepsilon = v_S = -\frac{r}{A}\sqrt{\frac{2M}{r}}, \tag{2.16}$$

where v_S is the fall velocity of an observer in the Kerr field and α_S the associated Lorentz factor. For further details, we refer again to [1].

The 5-dimensional Ricci can be calculated with the quantities (2.15), where $R_{ab} = 0$ applies due to the flatness of the embedding space. For the dimensional reduction, we collect all 0-terms on the right-hand side of the equation and obtain geometrically interpretable quantities for the stress-energy momentum tensor of the auxiliary model via the Einstein tensor.

As indicated above, rotational quantities already appear in the static auxiliary model. It is also noteworthy that the stress-energy-momentum tensor can be built up from such quantities via the Ricci. There are complicated relations between the 0-quantities and the rotational quantities, as noted in [1] and the mathematical appendix in [1]. After laborious conversions, we finally find[2]

$$R_{mn} = \begin{pmatrix} 2\tilde{H}_{13}\tilde{H}_{13} & & & \\ & -2\tilde{H}_{23}\tilde{H}_{23} & & \\ & & -\tilde{\Omega}_{rs}\tilde{\Omega}^{rs} - 2\tilde{F}_s\tilde{F}^s & \\ & & & \tilde{\Omega}_{rs}\tilde{\Omega}^{rs} + 2\tilde{F}_s\tilde{F}^s \end{pmatrix} \sin^2\varepsilon + 2F_{(m}E_{n)}, \qquad (2.17)$$

where $m = \{1, 2, ..., 3\}$. We have developed the static auxiliary model to such an extent that we can take the next step towards the actual Kerr model.

3. THE IMPLEMENTATION OF THE ROTATION

The essential step towards a rotating model is the introduction of an anholonomic transformation that contains the orbital velocity of the corotating observers. It operates on the bein vectors of the static metric

$$\overset{m}{e}_i = \Lambda_i^{i'} \overset{m}{e}_{i'}, \quad \underset{m}{e}^i = \Lambda_{i'}^{i} \underset{m}{e}^{i'}, \quad g_{ik} = \Lambda_{ik}^{i'k'} g_{i'k'} \qquad (3.1)$$

and has the form

$$\begin{aligned} &\Lambda_3^{3'} = \alpha_R, \quad \Lambda_4^{3'} = i\alpha_R\omega, \quad \Lambda_3^{4'} = -i\alpha_R\omega\sigma^2, \quad \Lambda_4^{4'} = \alpha_R \\ &\Lambda_{3'}^{3} = \alpha_R, \quad \Lambda_{4'}^{3} = -i\alpha_R\omega, \quad \Lambda_{3'}^{4} = i\alpha_R\omega\sigma^2, \quad \Lambda_{4'}^{4} = \alpha_R \end{aligned} \qquad (3.2)$$

with the quantities known from section 2. This leads to the metric

$$ds^2 = {dx^1}^2 + {dx^2}^2 + \left[\alpha_R dx^3 + i\alpha_R\omega\sigma dx^4\right]^2 + a_S^2\left[-i\alpha_R\omega\sigma dx^3 + \alpha_R dx^4\right]^2 \qquad (3.3)$$

which is much more expressive than the metric in Boyer-Lindquist form. The anholonomic differentials are given by

$$dx^1 = \alpha_S a_R dr, \quad dx^2 = \Lambda d\vartheta, \quad dx^3 = \sigma d\varphi, \quad dx^4 = \rho_S di\psi = idt. \qquad (3.4)$$

It is easy to convince oneself that the rotation is genuine, i.e., that it cannot be eliminated by a Lorentz transformation of the observer system. This is provided by the anholonomy of the transformation (3.2), specifically by

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[2] Note that the quantities without tildes include the factor a_s.

$$\Lambda^{j}_{[k'|i']} \neq x^{j}_{|[k'|i']} \neq 0 . \tag{3.5}$$

We define the tetrad object of anholonomy with

$$\Lambda_{mn}{}^{s} = \underset{m}{e}^{i'} \underset{n}{e}^{k'} \overset{s}{e}_{j'} \Lambda^{j'}_{j} \Lambda^{j}_{[k'|i']} \tag{3.6}$$

and thus arrive at the Ricci-rotation coefficients of Kerr geometry

$$A_{mn}{}^{s} = {}^{*}A_{mn}{}^{s} + \Omega_{mn}{}^{s}, \quad \Omega_{mns} = \Lambda_{mns} + \Lambda_{smn} - \Lambda_{nsm} . \tag{3.7}$$

Here, ${}^{*}A$ are the connexion coefficients of the static auxiliary model. Surprisingly, the emerging terms of the Ricci can be completely separated from those of the static model. We have

$$R_{mn} = {}^{*}R_{mn} + {}^{**}R_{mn} . \tag{3.8}$$

We have examined such a decomposition in general form in [2].

After treating the $\Omega_{mn}{}^{s}$ using (3.6) and (2.3), the following Maxwell-like equations are obtained

$$\begin{aligned} {}^{**}R_{33} &= -\left[F^{s}{}_{||s} - \Omega^{C}_{rs}\Omega^{sr}_{C} \right] \\ {}^{**}R_{34} &= \Omega^{s}_{C3||s} + 2F_{s}H^{3s}_{C} \\ {}^{**}R_{44} &= F^{s}{}_{||s} - \Omega^{C}_{rs}\Omega^{sr}_{C} \end{aligned} \tag{3.9}$$

with the quantities[3]

$$\begin{gathered} H^{3\alpha}_{C} = \left\{ 0, \tilde{H}_{32}\cos\varepsilon \right\}, \quad \Omega^{\alpha}_{C3} = \left\{ \tilde{H}_{13}, \tilde{H}_{23}\cos\varepsilon \right\} \\ \Omega^{C}_{rs}\Omega^{sr}_{C} = 2\left[\tilde{H}_{13}\tilde{H}_{13} - \tilde{H}_{23}\tilde{H}_{23}\cos^{2}\varepsilon \right] \end{gathered} \tag{3.10}$$

explained in more detail in [1].

The second part of the Ricci contains only rotational field strengths. Solving the equations (3.9) first leads to ${}^{**}R_{34} = 0$, while the rest of the rotational Ricci gives the relation (2.17) with a negative sign, so that (3.8) is fulfilled and, as expected, a vacuum solution is provided.

4. CONCLUSIONS

We showed that the Ricci of the Kerr model can be divided into two parts, one corresponding to the static part of the Kerr metric and the second to the rotational part. This enhances the understanding of the Kerr model. However, the method could still help find the interior Kerr metric, which has been sought for years. For the auxiliary static model of an interior solution, we have proposed the cap of an ellipsoid [3], whose stress-energy momentum tensor is conserved. Hopefully, this approach can be extended to a complete model via anholonomic transformation. A suitable differential rotation law would have to be

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[3] The somewhat seemingly strange structures result from merging the quantities D with H according to (2.10) and (2.11).

found whose orbital velocity follows the exterior solution but decreases towards the interior and vanishes at the axis of rotation.

5. REFERENCES

[1] Burghardt R., *Spacetime curvature*. 2024, http://arg.or.at/EMono.htm

Raumkrümmung. 2024, http://arg.or.at/Mono.htm

[2] Burghardt R., *Anholonomy and gravitation.* http://www.arg.or.at/Wpdf/WAnh.pdf

[3] Burghardt R., *The interior of an elliptic stellar object*. http://arg.or.at/Wpdf/WElp.pdf